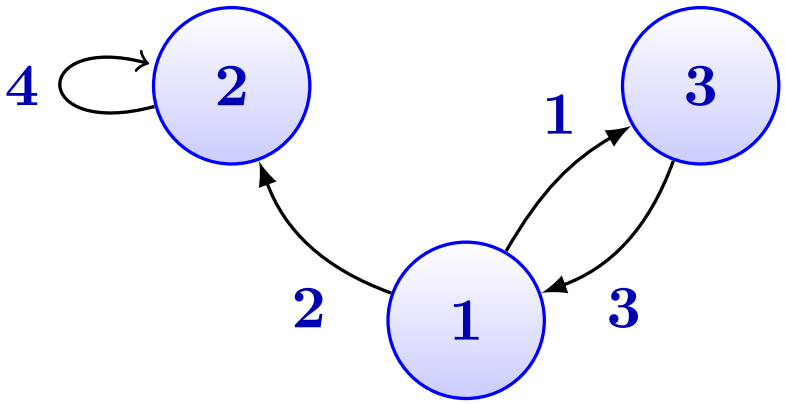
Sparse graphs using exchangeable random measures

François Caron

Department of Statistics, Oxford

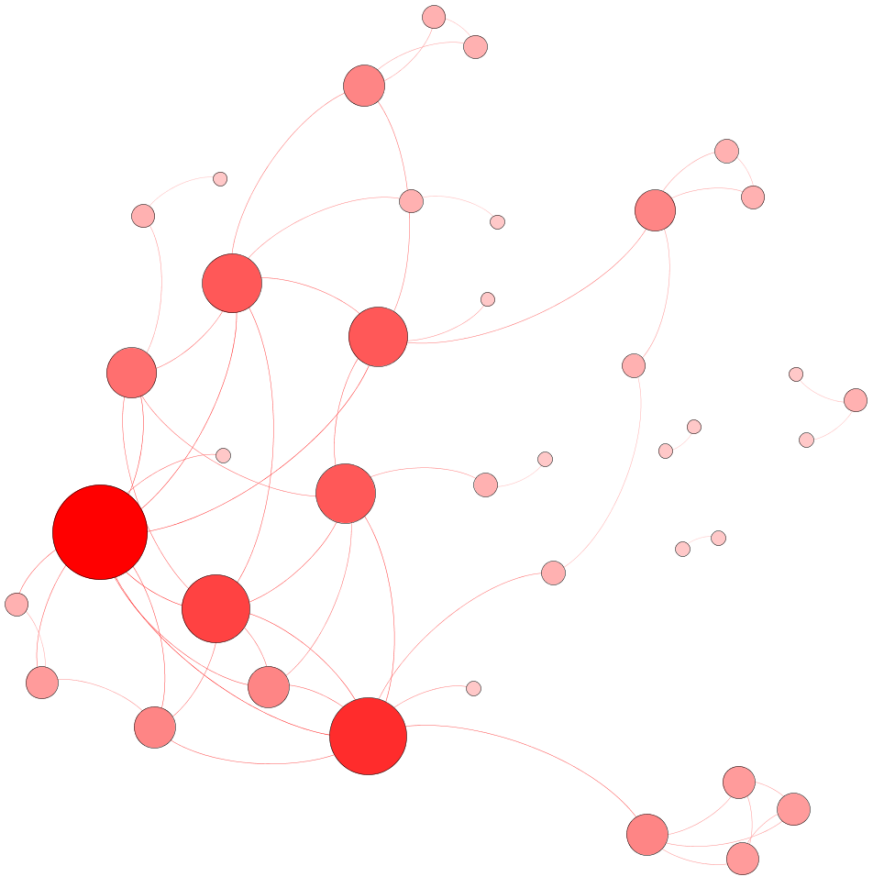
Oxford/Warwick workshop
Scalable Statistical Methods for Analysis of large and complex data sets
October 9, 2015

Introduction



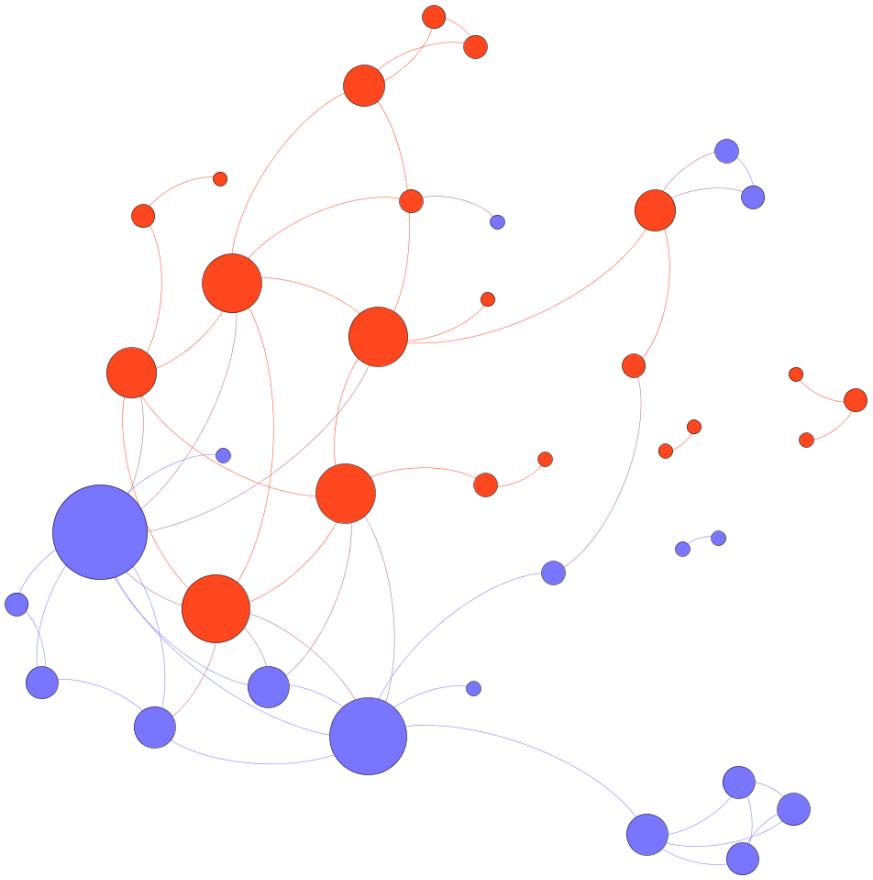
- Directed Multigraphs
  - Emails
  - Citations
  - WWW

Introduction



- Simple graphs
  - Social network
  - Protein-protein interaction

Introduction



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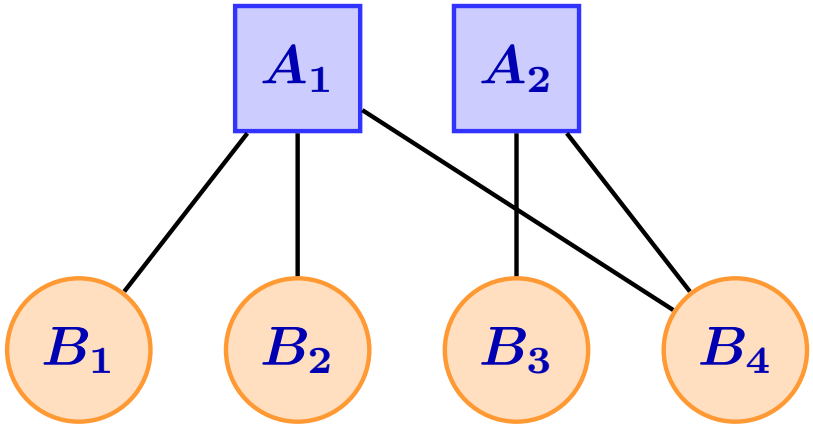
Introduction



- Bipartite graphs
  - Scientists authoring papers
  - Readers reading books
  - Internet users posting messages on forums
  - Customers buying items

Introduction

- Build a statistical model of the network to
  - Find interpretable structure in the network
  - Predict missing edges
  - Predict connections of new nodes

Introduction

- Massive networks
  - Linkedin: $\simeq$ 300 millions
  - Facebook: $\simeq$ billion
  - Twitter: $\simeq$ 300 millions
  - www: $\simeq$ billion
- Capture large-scale properties of networks
- Scalable inference algorithms

Introduction

- Properties of real-world networks
  - Sparsity

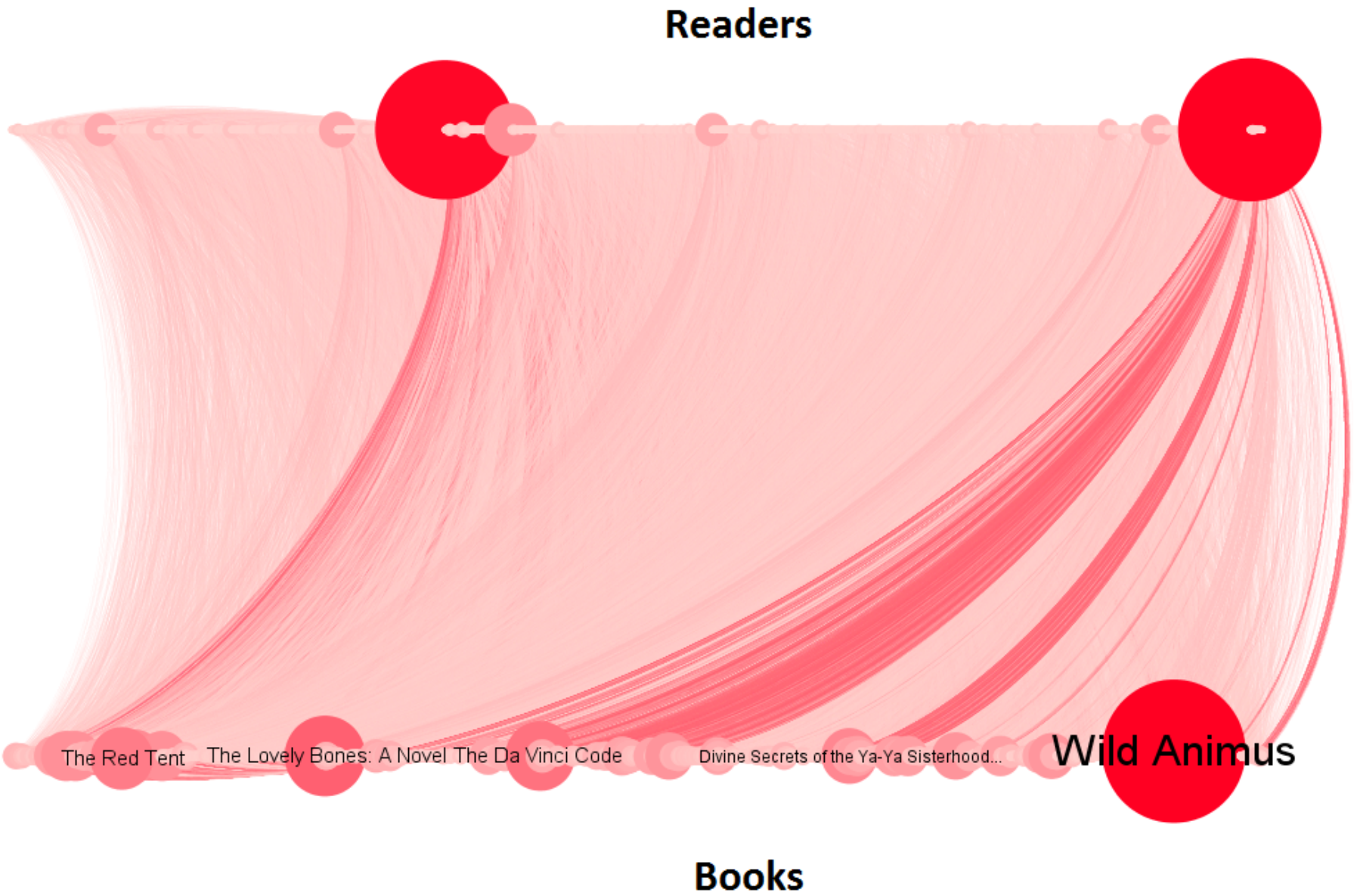
    Dense graph: $n_e = \Theta(n^2)$
    Sparse graph: $n_e = o(n^2)$

    with n_e the number of edges and n the number of nodes
  - Heavy-tailed degree distributions
  - Latent structure

[Newman, 2009, Clauset et al., 2009]

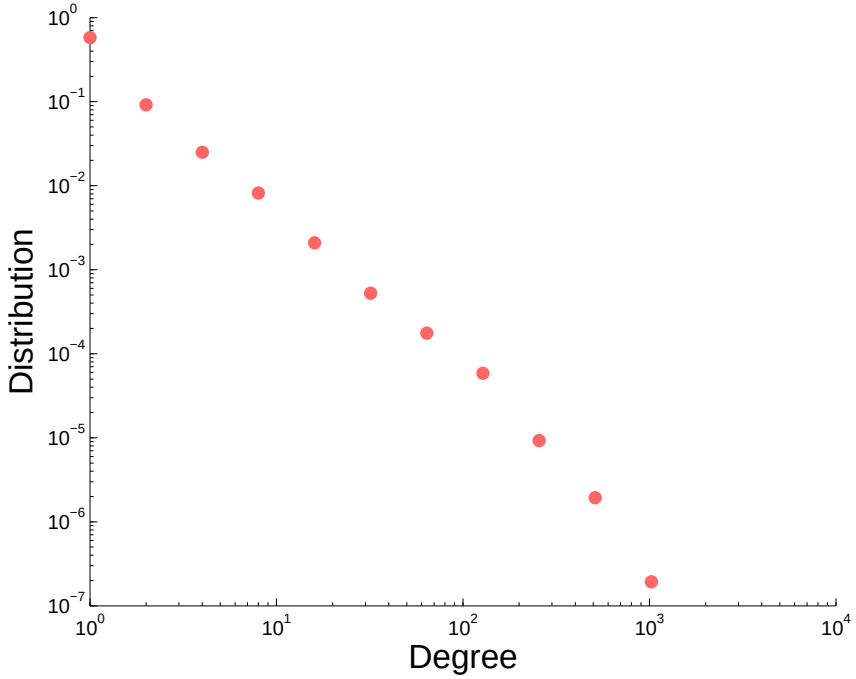
Book-crossing community network

5 000 readers, 36 000 books, 50 000 edges

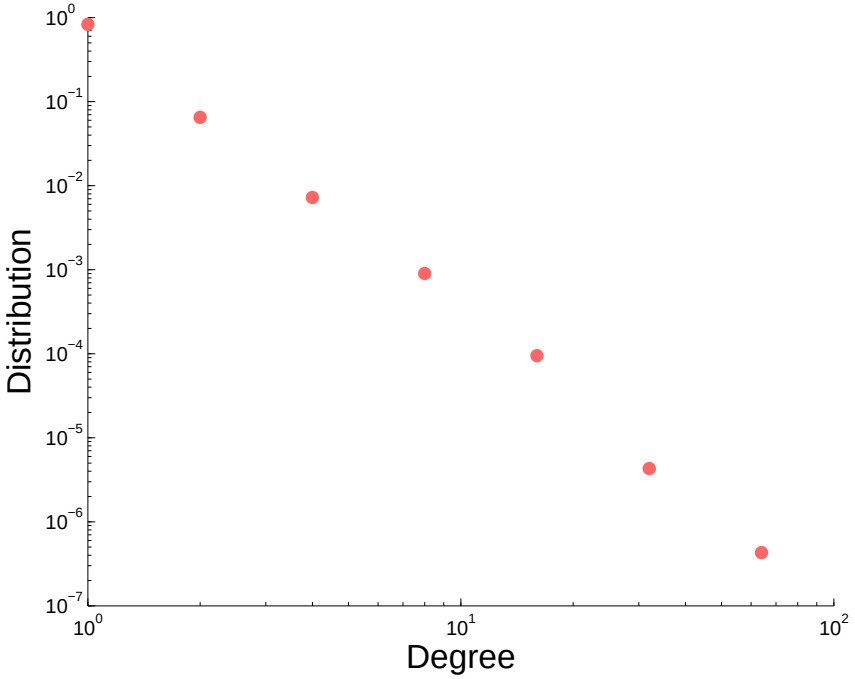


Book-crossing community network

Degree distributions on log-log scale



(a) Readers



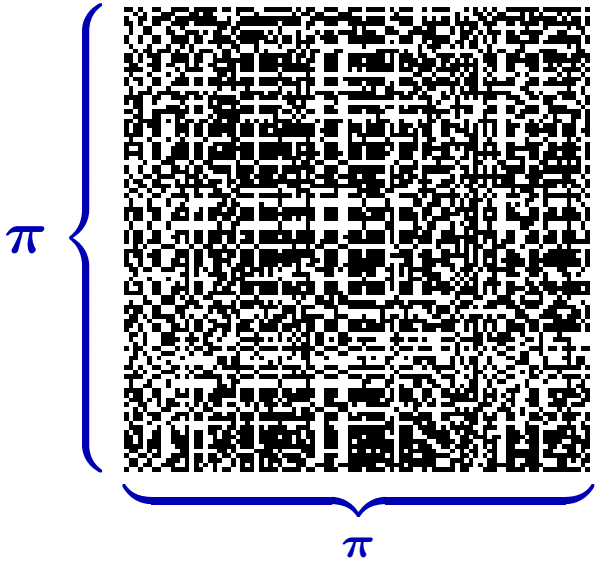
(b) Books

Introduction

- Simple graphs
- Adjacency matrix $X_{ij} \in \{0,1\}, (i,j) \in \mathbb{N}^2$
- Joint exchangeability

$$(X_{ij}) \stackrel{d}{=} (X_{\pi(i)\pi(j)})$$

for any permutation π of $\mathbb{N}$



Introduction

- Aldous-Hoover representation theorem for exchangeable binary matrices
$$\boldsymbol{X_{ij}|U_i, U_j, W \sim \mathrm{Ber}(W(U_i, U_j))}$$
with $\boldsymbol{U_i} \overset{\text{iid}}{\sim} \mathbf{Unif(0,1)}$ and $\boldsymbol{W} : \mathbf{[0,1]^2 \to [0,1]}$ a random function
- Several network models fit in this framework
  - Erdös-Rényi, (mixed-membership) stochastic block-models, infinite relational models, etc

[Hoover, 1979, Aldous, 1981, Lloyd et al., 2012, Orbanz and Roy, 2015]

Introduction

- Corollary of A-H theorem

  Graphs represented by an exchangeable matrix are either trivially empty or dense

- To quote the survey paper of Orbanz and Roy

  "the theory [...] clarifies the limitations of exchangeable models. It shows, for example, that most Bayesian models of network data are inherently misspecified"

[Hoover, 1979, Aldous, 1981, Lloyd et al., 2012, Orbanz and Roy, 2015]

Introduction

How to handle sparse graphs?

- Give up infinite exchangeability?
  - Non-exchangeable generative models
    - Preferential attachment model
  - Sequence of finitely exchangeable models $(X_{ij}^{(n)})_{1 \leq i,j \leq n}$
    - Chung-Lu

$$X_{ij}^{(n)} \sim \mathrm{Ber}\left(\frac{w_i w_j}{\sum_{k=1}^{n} w_k}\right)$$

    - Sparsification of the graphon

$$X_{ij}^{(n)} \sim \mathrm{Ber}(\rho_n W(U_i, U_j))$$

  with $\rho_n \to 0$

[Barabási and Albert, 1999, Chung and Lu, 2002, Bickel and Chen, 2009, Wolfe and Olhede, 2013]

Point process representation

- Representation of a graph as a (marked) point process over $\mathbb{R}_+^2$
- Representation theorem by Kallenberg for jointly exchangeable point processes on the plane
- Construction based on completely random measures
- Properties of the model
  - Exchangeable point process
  - Sparsity
  - Heavy-tailed degree distributions
- Scalable inference

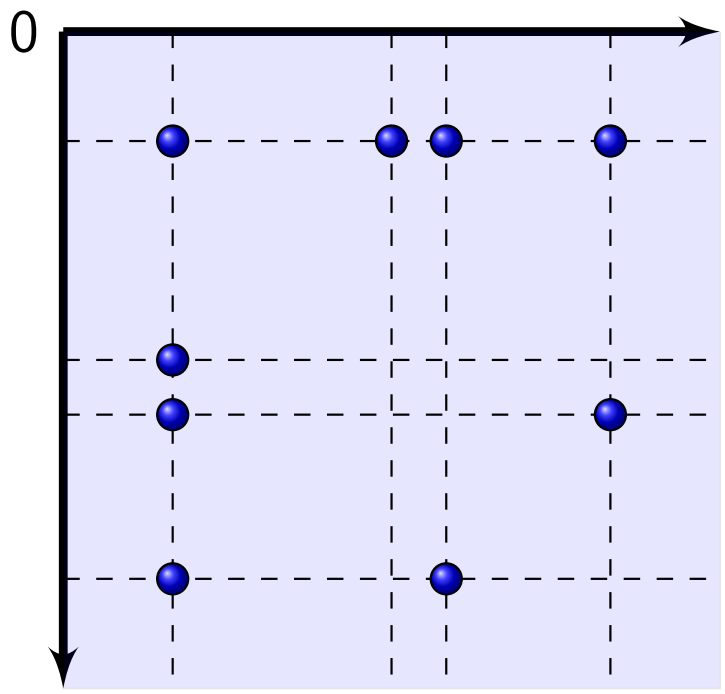
[Kallenberg, 2005, Caron and Fox, 2014]

Point process representation

- Undirected graph represented as a point process on $\mathbb{R}_+^2$

$$Z = \sum_{i,j} z_{ij} \delta_{(\theta_i, \theta_j)}$$

with $\theta_i \in \mathbb{R}_+$, $z_{ij} \in \{0, 1\}$ with $z_{ij} = z_{ji}$



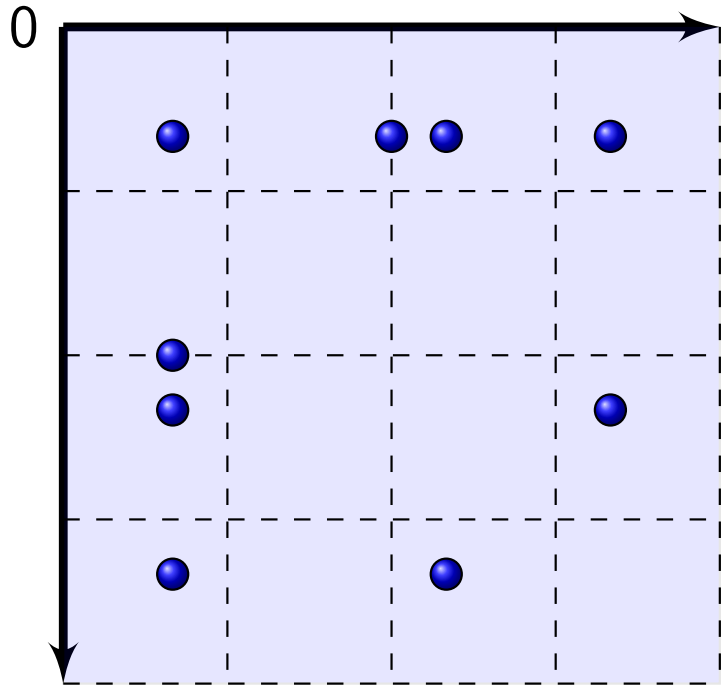
Point process representation

Joint exchangeability

Let $A_i = [h(i-1), hi]$ for $i \in \mathbb{N}$ then

$$(Z(A_i \times A_j)) \stackrel{d}{=} (Z(A_{\pi(i)} \times A_{\pi(j)}))$$

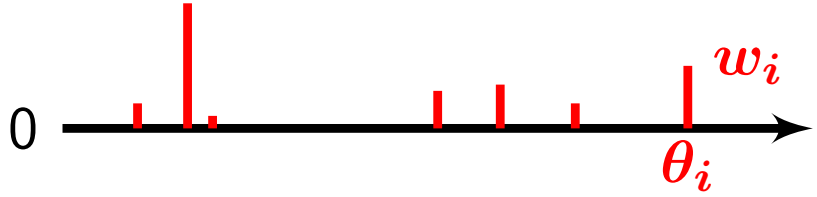
for any permutation π of $\mathbb{N}$ and any $h > 0$



Completely random measures

- Nodes are embedded at some location $\theta_i \in \mathbb{R}_+$
- Each node has a sociability parameter w_i
- Homogeneous completely random measure on $\mathbb{R}_+$

$$W = \sum_{i=1}^{\infty} w_i \delta_{\theta_i} \quad W \sim \text{CRM}(\rho, \lambda).$$



- Lévy measure $\nu(dw, d\theta) = \rho(dw)\lambda(d\theta)$

$$\int_0^{\infty} \rho(dw) = \infty \Longrightarrow \text{Infinite number of jumps in any interval } [0, T]$$

$$\int_0^{\infty} \rho(dw) < \infty \Longrightarrow \text{Finite number of jumps in any interval } [0, T]$$

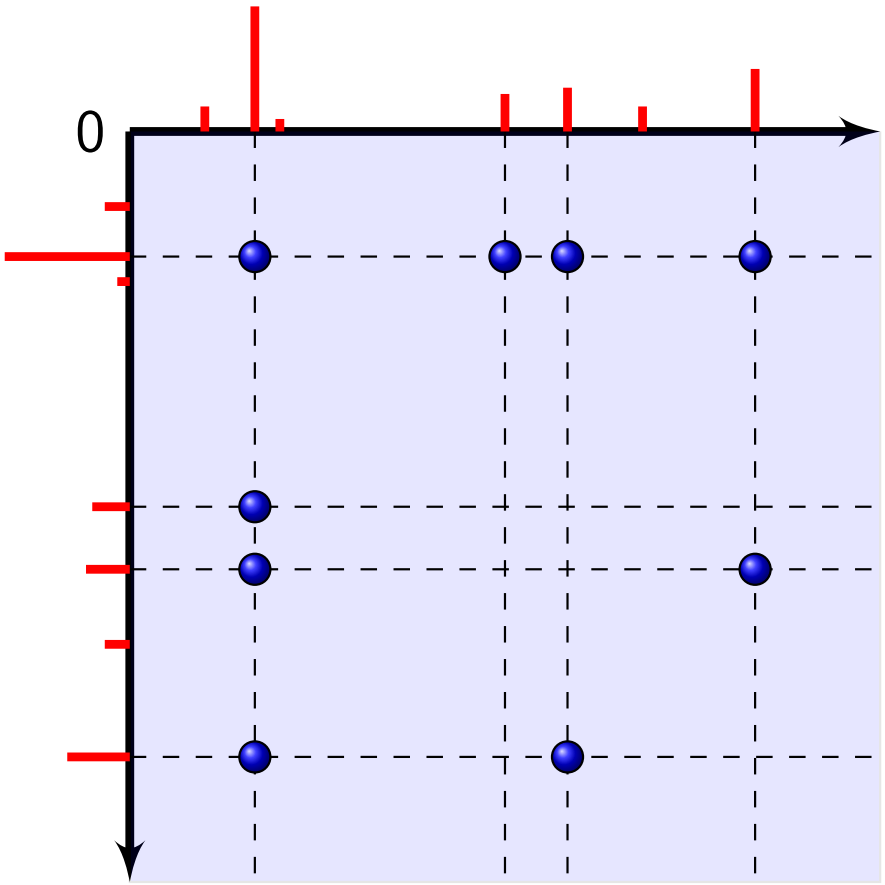
[Kingman, 1967]

Model for undirected graphs

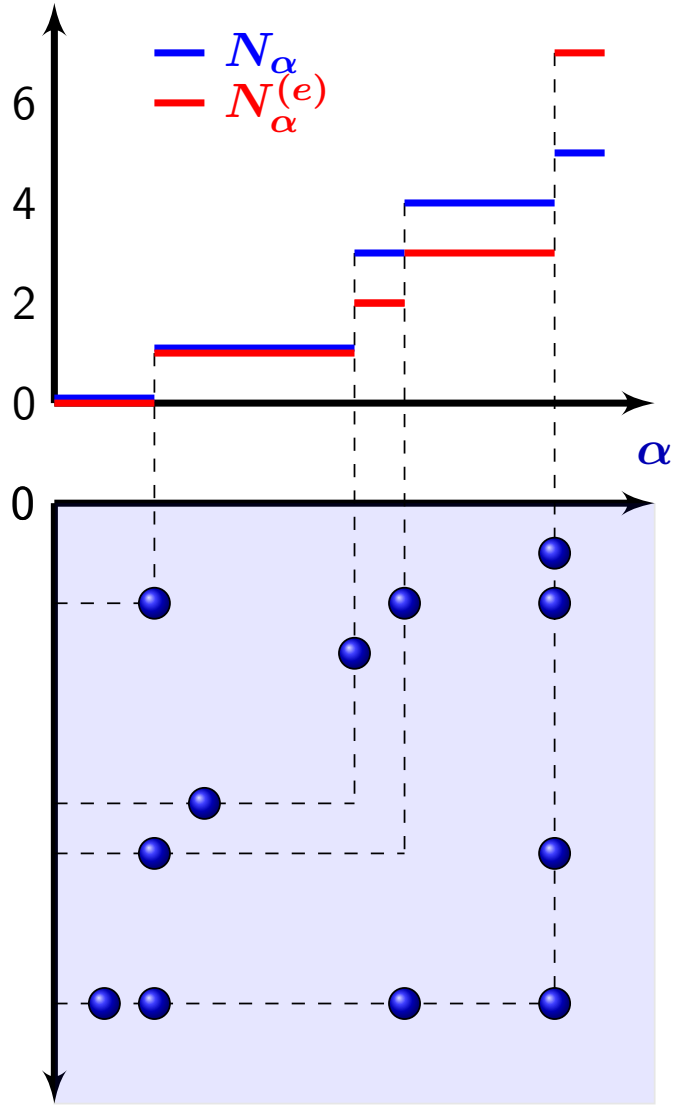
- For $i \leq j$

$$\Pr(z_{ij} = 1 \mid w) = \begin{cases} 1 - \exp(-2w_i w_j) & i \neq j \\ 1 - \exp(-w_i^2) & i = j \end{cases}$$

and $z_{ji} = z_{ij}$



Properties: Sparsity



Properties: Sparsity

Assume $\rho \neq 0$ and $\mathbb{E}[W([0,1])] < \infty$.

Theorem

Let N_α be the number of nodes and $N_\alpha^{(e)}$ the number of edges in the undirected graph restriction, Z_α. Then

$$N_\alpha^{(e)} = \begin{cases} \Theta\left(N_\alpha^2\right) & \text{if } W \text{ is finite-activity} \\ o\left(N_\alpha^2\right) & \text{if } W \text{ is infinite-activity} \end{cases}$$

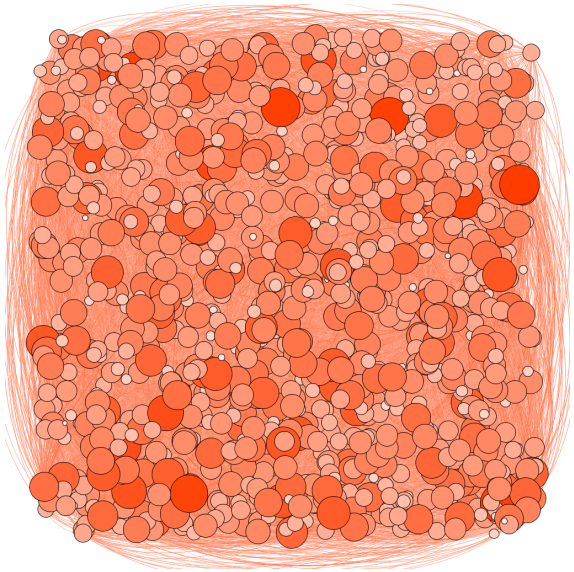
almost surely as $\alpha \to \infty$.

Particular case: Generalized Gamma Process

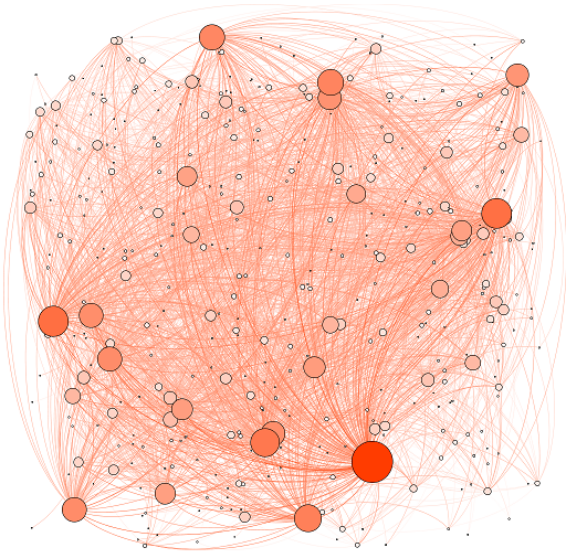
- Lévy intensity
$$\frac{1}{\Gamma(1-\sigma)} w^{-1-\sigma} e^{-\tau w}$$
with $\sigma \in (-\infty, 0]$ and $\tau > 0$
or $\sigma \in (0, 1)$ and $\tau \geq 0$
- Infinite activity for $\sigma \geq 0$
- Exact sampling of the graph via an urn process
- Power-law degree distribution

[Brix, 1999, Lijoi et al., 2007]

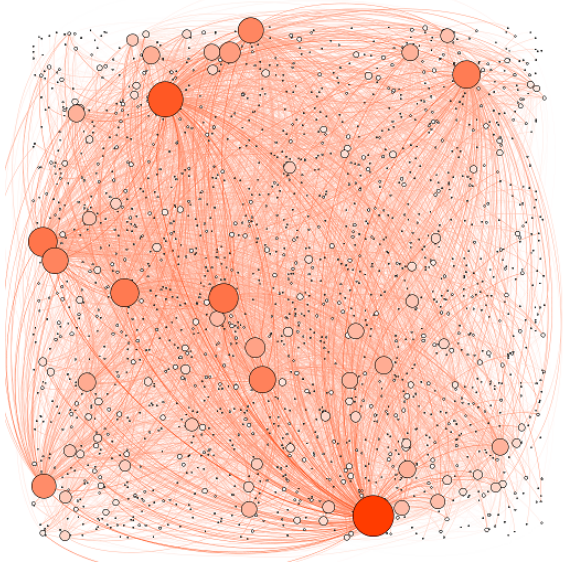
Particular case: Generalized Gamma Process



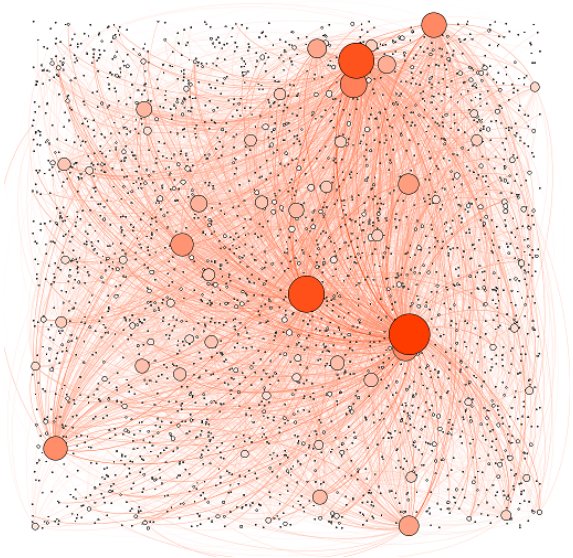
Erdös-Rényi $G(1000, 0.05)$



Gamma Process



GGP ($\sigma = 0.5$)

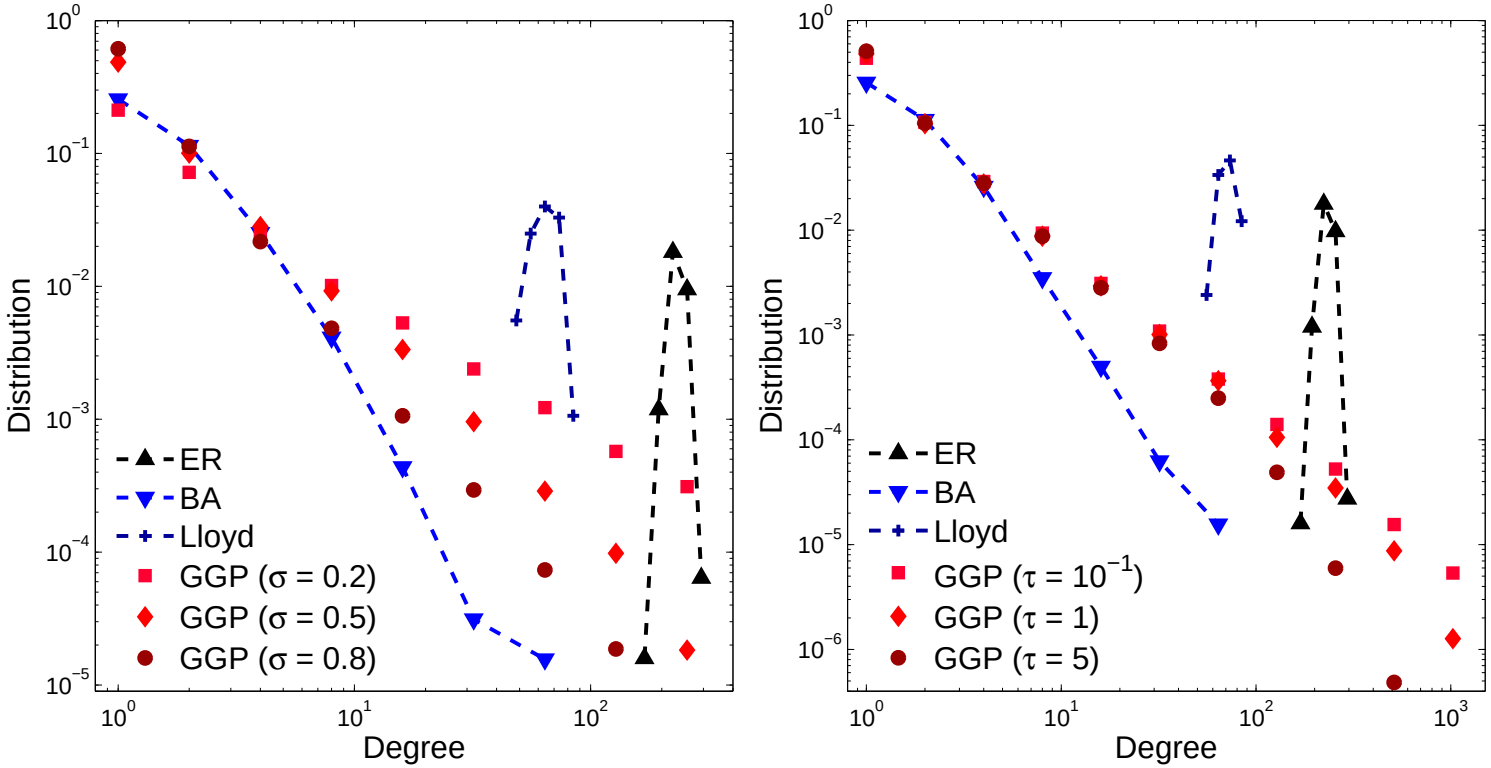


GGP ($\sigma = 0.8$)

Particular case: Generalized Gamma Process

Power-law degree distributions

- Power-law like behavior providing a heavy-tailed degree distribution
- Higher power-law exponents for larger σ
- The parameter τ tunes the exponential cut-off in the tails.



Particular case: Generalized Gamma Process

Posterior inference

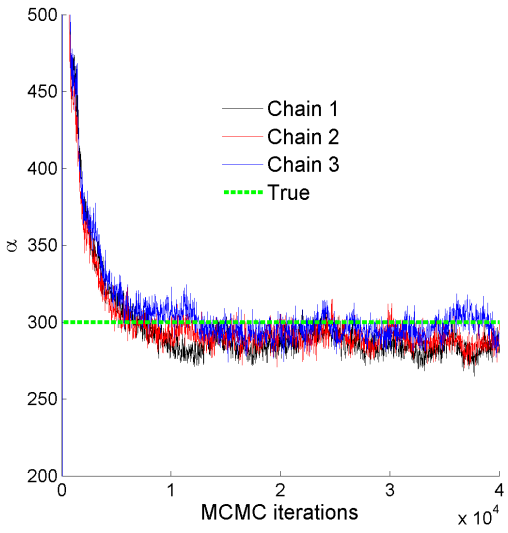
- Let $\phi = (\alpha, \sigma, \tau)$ with improper priors
- We want to approximate

$$p(w_1, \ldots, w_{N_\alpha}, w_*, \phi | (z_{ij})_{1 \leq i,j \leq N_\alpha})$$

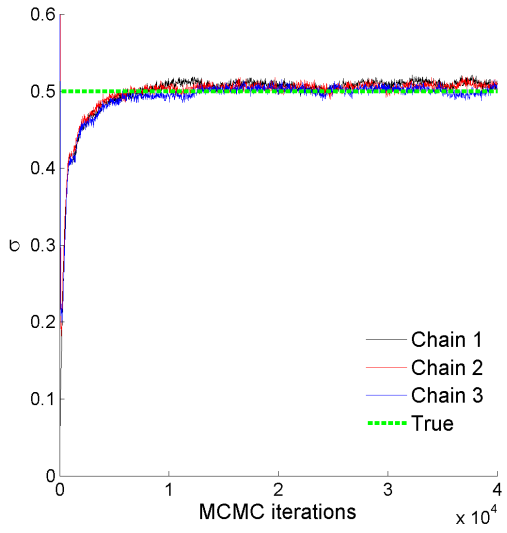
- Latent count variables $\overline{n}_{ij} = n_{ij} + n_{ji}$
- Markov chain Monte Carlo sampler
  1. Update the weights $(w_1, \ldots, w_{N_\alpha})$ given the rest using an Hamiltonian Monte Carlo update
  2. Update the total mass w_* and hyperparameters $\phi = (\alpha, \sigma, \tau)$ given the rest using a Metropolis-Hastings update
  3. Update the latent counts $(\overline{n}_{ij})$ given the rest from a truncated Poisson distribution

Simulated data

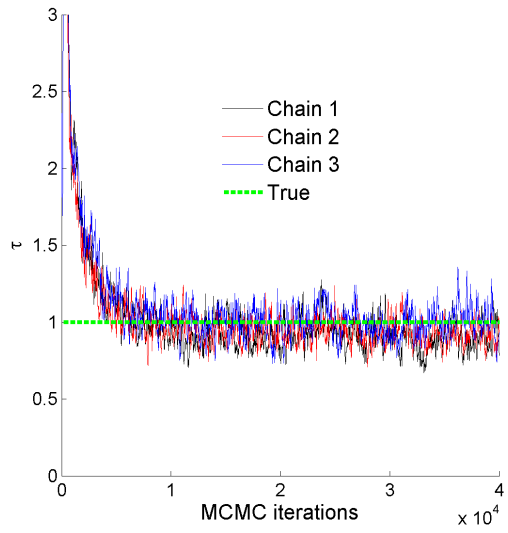
- Simulation of a GGP graph with $\boldsymbol{\alpha = 300, \sigma = 1/2, \tau = 1}$
- 13,995 nodes and 76,605 edges
- MCMC sampler with 3 chains and 40,000 iterations
- Takes 10min on a standard desktop with Matlab



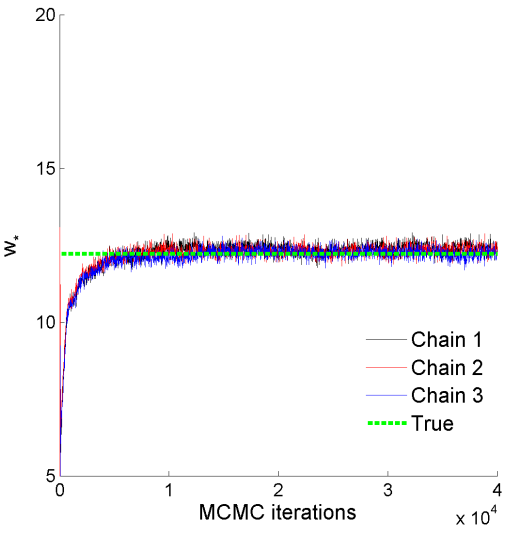
(a) α



(b) σ

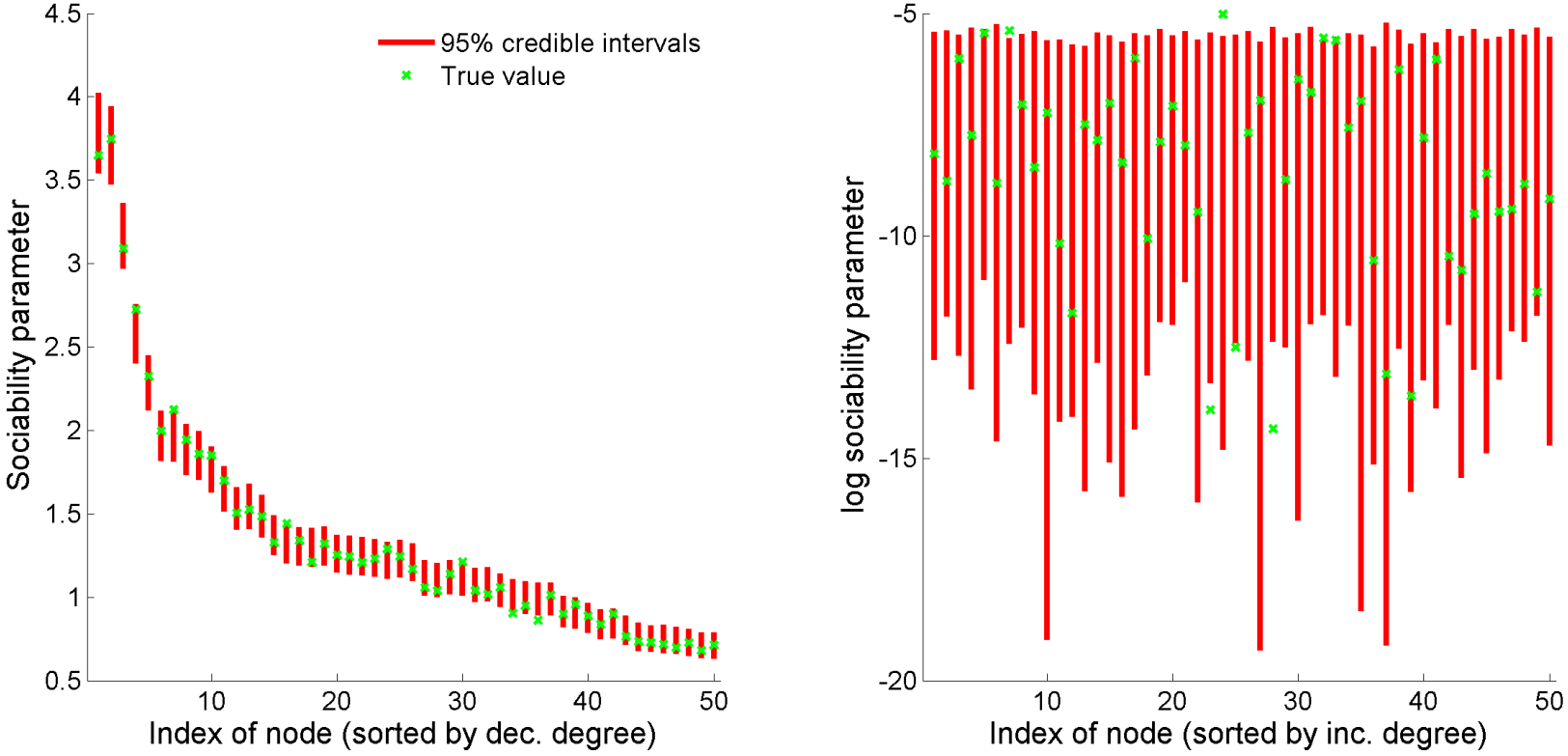


(c) τ



(d) w_*

Simulated data



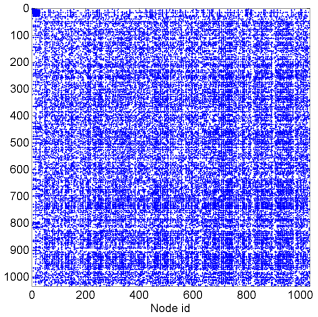
(a) 50 nodes with highest degree

(b) 50 nodes with lowest degree

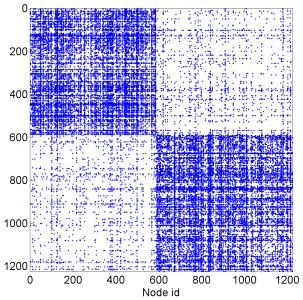
Figure: 95 % posterior intervals of (a) the sociability parameters w_i of the 50 nodes with highest degree and (b) the log-sociability parameter $\log w_i$ of the 50 nodes with lowest degree. True values are represented by a green star.

Real network data

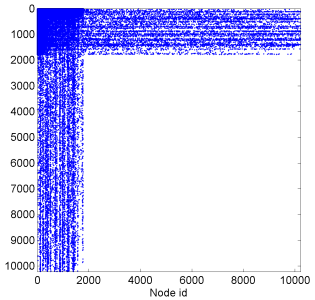
- Assessing the sparsity of the network
- We aim at reporting $\mathbf{Pr}(\boldsymbol{\sigma} \geq \mathbf{0}|\boldsymbol{z})$ based on a set of observed connections $(\boldsymbol{z})$
- 12 different networks
- $\sim \mathbf{1,000 - 300,000}$ nodes and $\mathbf{10,000 - 1,000,000}$ edges



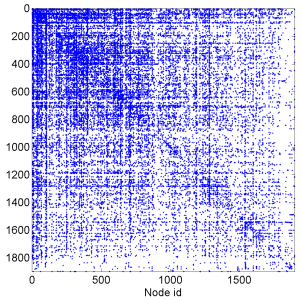
(a) facebook107



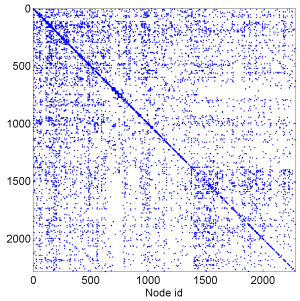
(b) polblogs



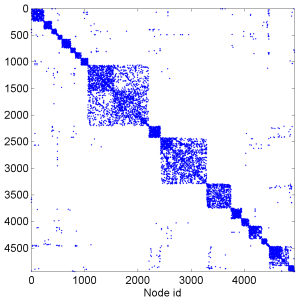
(c) USairport



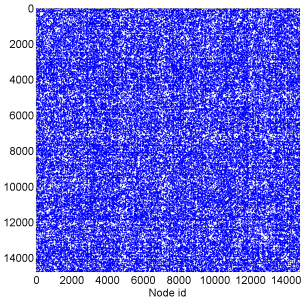
(d) UCirvine



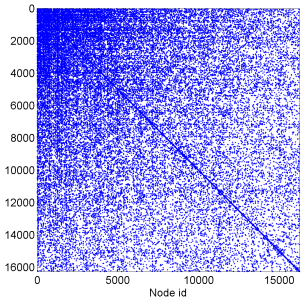
(e) yeast



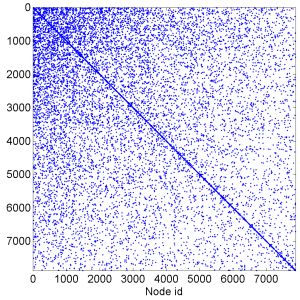
(f) USpower



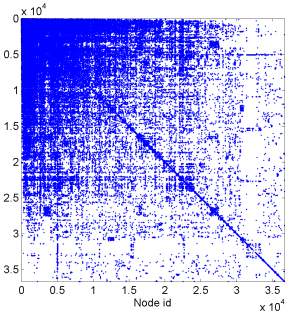
(g) IMDB



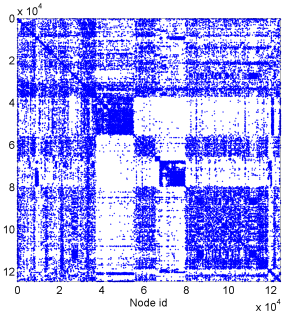
(h) cond-mat1



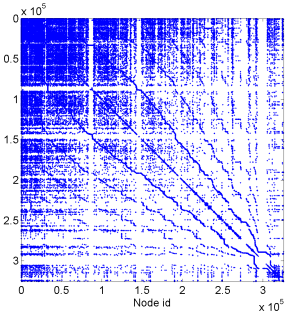
(i) cond-mat2



(j) enron



(k) internet



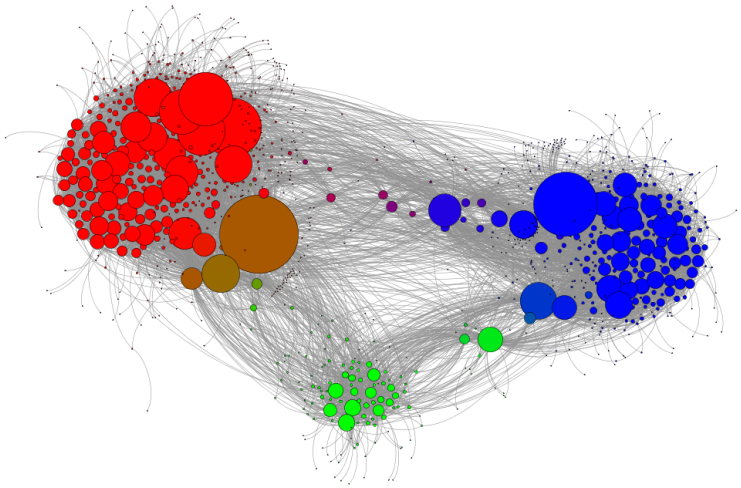
(l) www

Real network data

Name	Nb nodes	Nb edges	Time (min)	$\mathbf{Pr}(\boldsymbol{\sigma} \geq \mathbf{0}\|\boldsymbol{z})$	99% CI $\boldsymbol{\sigma}$
facebook107	1,034	26,749	1	0.00	$[-1.06, -0.82]$
polblogs	1,224	16,715	1	0.00	$[-0.35, -0.20]$
USairport	1,574	17,215	1	1.00	$[\ 0.10,\ 0.18]$
UCirvine	1,899	13,838	1	0.00	$[-0.14, -0.02]$
yeast	2,284	6,646	1	0.28	$[-0.09, 0.05]$
USpower	4,941	6,594	1	0.00	$[-4.84, -3.19]$
IMDB	14,752	38,369	2	0.00	$[-0.24, -0.17]$
cond-mat1	16,264	47,594	2	0.00	$[-0.95, -0.84]$
cond-mat2	7,883	8,586	1	0.00	$[-0.18, -0.02]$
Enron	36,692	183,831	7	1.00	$[\ 0.20,\ 0.22]$
internet	124,651	193,620	15	0.00	$[-0.20, -0.17]$
www	325,729	1,090,108	132	1.00	$[0.26, 0.30]$

Conclusion

- Statistical network models
- Build on exchangeable random measures
- Sparsity and power-law properties
- Scalable inference
- Extensions to more structured models: non-negative factorization, block-model, covariates, dynamic networks,etc



- Matlab code available

  http://www.stats.ox.ac.uk/~caron/code/bnpgraph/

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